CS3621 Quiz 1 (Fall 2005) Solutions

1. Do the following problems. Only writing down your result is sufficient.

   (a) **[8 points]** Rewrite the following second degree equation, which represents a quadric surface, to its equivalent matrix form:

   $$3x^2 + 2y^2 - 4z^2 - 4xy + 10xz - 20yz + 10x - 8y + 12z + 1 = 0$$

   Solution: The following is the answer:

   $$\begin{bmatrix} 3 & -2 & 5 & 5 \\ -2 & 2 & -10 & -4 \\ 5 & -10 & -4 & 6 \\ 5 & -4 & 6 & 1 \end{bmatrix}$$

   (b) **[8 points]** Convert the following equation to a homogeneous form:

   $$(x^2 + y^2)^3 + xy(x^2 + xyz) + x^2y^2z^2 + 1 = 0$$

   Solution: The following is the answer:

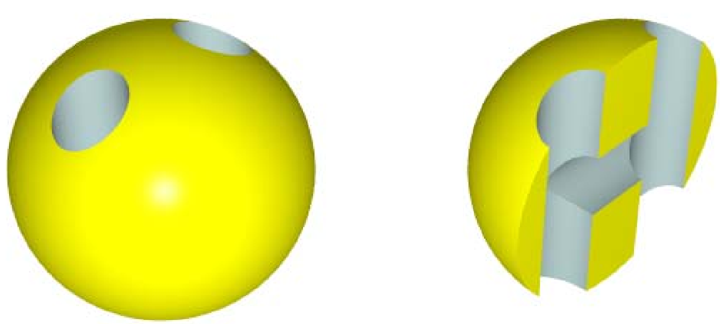
   $$(x^2 + y^2)^3 + xyw(x^2w + xyz) + x^2y^2z^2 + w^6 = 0$$

   (c) **[8 points]** We know the following equation represents a conic section (*i.e.*, ellipse, hyperbola or parabola). Which conic section is it? Show your computation and reasoning. Only providing an answer will receive **no** credit.

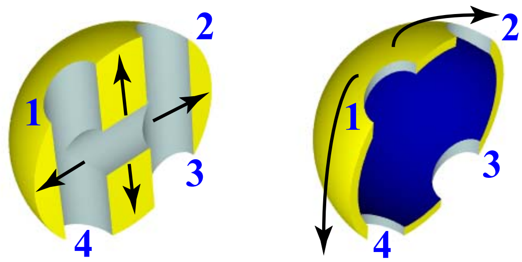
   $$2x^2 + 5xy + 6y^2 - 4x - 5y + 100 = 0$$

   Solution: Here we have $B = 5/2$, $A = 2$ and $C = 6$. Since $B^2 - A \cdot C = (5/2)^2 - 2 \cdot 6 = -5.75 < 0$, this is an ellipse.

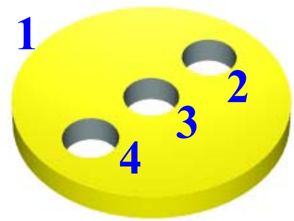
2. **[12 points]** Suppose a sphere is punched by a H-shape tunnel as shown in the following figures. What is the genus value G of this model? You should elaborate how you obtain your result. Only providing a number receives no credit.



Solution: With the technique discussed on the Euler-Poincaré formula page, it is easy to see that the genus number is $G = 3$. First, we push the interior of the sphere toward its shell as shown in the left figure below.



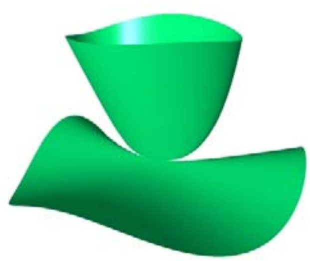
Then, we stretch hole 1 (above right) and collapse the shell on a plane. The result should look like the figure below:



Now, it is easy to see that the number of penetrating holes is 3, and, hence, $G = 3$.

3. **[9 points]** The following is a part of a surface. The two portions shown are tangent to each other. Is it a 2-manifold? Elaborate your finding. Only providing an answer receives **no** credit.

   Solution: This is not a 2-manifold because any open ball with the center at the "tangent point" intersects the surface in *two* sheets that are tangent to each other. As a result, they cannot be transformed to a single sheet without cutting and gluing.

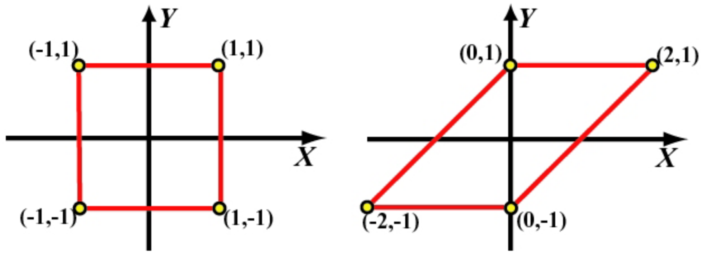


4. **[15 points]** In a modeling practice, you are to transform a square in the xy-plane defined by (1,1), (-1,1), (-1,-1) and (1,-1) to a parallelogram whose corresponding vertices are (2,1), (0,1), (-2,-1) and (0,-1). What is the transformation matrix that can achieve this goal? Show me how you obtain this transformation matrix. Only providing an answer receives **no** credit. I will enforce this rule strictly. The following figure shows the required transformation.

   Solution: There are at least three possible ways to solve this problem.

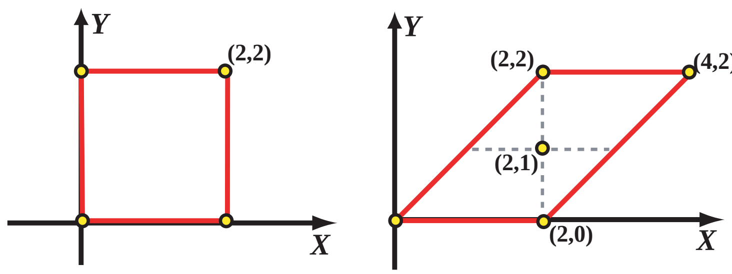
   Method 1: If you are familiar with the shear transformation, this problem is very simple. Point $(1, 1)$ is mapped to $(2, 1)$, which means every point (x, y) is "pushed" in the direction of the x-axis to $(x + y, y)$. Thus, the transformation matrix is simply:

$$\mathbf{S} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



It is easy to verify that $(1,1)$, $(1,-1)$, $(-1,-1)$ and $(-1,1)$ are mapped to $(2,1)$, $(0,-1)$, $(-2,-1)$ and $(0,1)$, respectively.

Method 2: Some many not be comfortable in transforming points with negative coordinate values. If this is the case, we can do it differently using all positive coordinate values. We can translate the lower-left corner $(-1,-1)$ to the origin (see below). After this translation, the upper-right corner is $(2,2)$. Then, a shear is applied to this new square. Since $(2,2)$ maps to $(4,2)$, (x,y) is mapped to $(x+y,y)$. Since the center of the parallelogram is $(2,1)$, a translation in the direction of $(-2,-1)$ will position the center of the parallelogram at the origin.



The first translation $\mathbf{T}_1$, shear $\mathbf{S}$ and the second translation $\mathbf{T}_2$ are:

$$\mathbf{T}_1 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \qquad \mathbf{S} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \mathbf{T}_2 = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

Therefore, the desired transformation matrix is

$$\mathbf{T}_2 \cdot \mathbf{S} \cdot \mathbf{T}_1 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

It is identical to the one obtained by the first method. Note the order of matrix multiplication.

Method 3: Since the desired transformation maps line segments to line segments and changes the shape of the object, it is an affine transformation. Let this affine transformation be

$$\mathbf{A} = \begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix}$$

Since $(1,1)$, $(1,-1)$, $(-1,-1)$ and $(-1,1)$ are mapped to $(2,1)$, $(0,-1)$, $(-2,-1)$ and $(0,1)$, respectively, we have

$$\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \mathbf{A} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \qquad \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} = \mathbf{A} \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} = \mathbf{A} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

From the first equation, we have

$$\begin{aligned} 2 &= a+b+c & (1) \\ 1 &= d+e+f & (2) \end{aligned}$$

From the second equation, we have

$$\begin{aligned} -2 &= -a-b+c & (3) \\ -1 &= -d-e+f & (4) \end{aligned}$$

From the third equation above, we have

$$\begin{aligned} 0 &= a-b+c & (5) \\ -1 &= d-e+f & (6) \end{aligned}$$

Adding Eqn (1) and Eqn (3) together, we have $c = 0$. Similarly, adding Eqn (2) and Eqn (4) together yields $f = 0$.

Subtracting Eqn (5) from Eqn (1) gives $b = 1$. Since $b = 1$ and $c = 0$, from either Eqn (1), Eqn (3) of Eqn (5) we have $a = 1$. Subtracting Eqn (6) from Eqn (2) gives $e = 1$. Substituting $e = 1$ and $f = 0$ into Eqn (2), Eqn (4) or Eqn (6), we have $d = 0$. Therefore, the desired transformation matrix is the one we computed earlier.

5. **[15 points]** A *winged-edge* representation consists of three tables, a vertex table, an edge table and a face table. Given a face f, actually an index to the face table, find all incident vertices of face f in clock-wise or counterclock-wise order. More precisely, design an algorithm *ListVertice*(f: face) that takes a face f and prints out all of its incident vertices in clock-wise or counterclock-wise order. You do not have to write a complete C/C++ program. Pseudo-code is perfectly fine; however, you should comment on your algorithm.

   Solution: The following is a possible solution:

```
e := faceTable[f];     // retrieve an incident edge of face f
s := e;       // save edge e to s for later use
repeat
   if f = e's left face then    // left and right faces would make a big difference
      print e's start vertex;    // take the start vertex
      e := e's left successor;    // take the left successor
   else      // if f is the right face of e
      print e's end vertex;    // take the end vertex
      e := e's right successor;    // take the right successor
   end if
until s = e;    // done if return to the initial edge
```

You may use Bryan's winged-edge visualization/animation program to verify and learn the above algorithm.