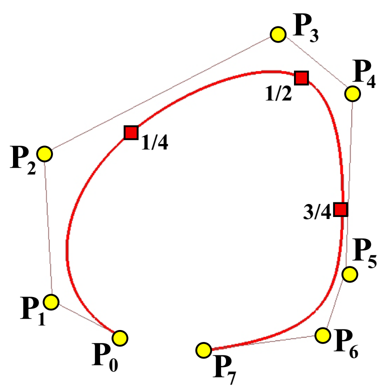
CS3621 Quiz 2 Solutions (Fall 2005)
75 points

1. **B-splines**
Suppose we have a *clamped* B-spline curve of degree 4 defined by 8 control points $\mathbf{P}_0$ to $\mathbf{P}_7$ and 13 knots 0, 0, 0, 0, 0, 1/4, 1/2, 3/4, 1, 1, 1, 1, 1. Note that 0 and 1 are multiple knots of multiplicity 5. The red squares on the curve mark the points corresponding to knots 1/4, 1/2 and 3/4.

   (a) **[10 points]** What is the convex hull that contains the curve segment defined on knot span $[1/4, 1/2)$ according to the strong convex hull property? Mark the convex hull directly in the figure and elaborate your answer. **You will receive <u>no</u> credit if you do not mark the result correctly or do not provide an elaboration.**

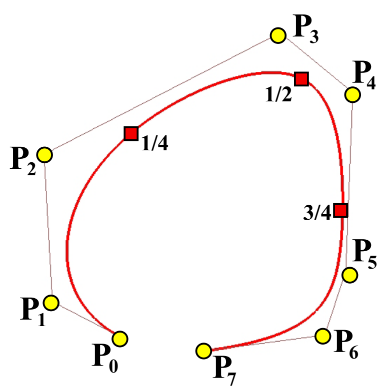


<u>Answer</u>: The following table shows the knot numbering:

$u_0 = u_1 = u_2 = u_3 = u_4$	u_5	u_6	u_7	$u_8 = u_9 = u_{10} = u_{11} = u_{12}$
0	1/4	1/2	3/4	1

Since the curve segment defined on $[u_i, u_{i+1})$ is contained in the convex hull of $p+1$ control points: $\mathbf{P}_{i-p}$, $\mathbf{P}_{i-p+1}$, ..., $\mathbf{P}_i$, the convex hull that contains the curve segment defined on $[1/4, 1/2) = [u_5, u_6)$ is defined by control points $\mathbf{P}_1$, $\mathbf{P}_2$, $\mathbf{P}_3$, $\mathbf{P}_4$ and $\mathbf{P}_5$. ∎

   (b) **[10 points]** If control point $\mathbf{P}_5$ is moved to a new position, which curve segments will be affected? Mark the curve segments directly in the figure and elaborate your answer. **You will receive <u>no</u> credit if you do not mark the result correctly or do not provide an elaboration.**



Answer: Refer to the knot table in the previous sub-problem. Changing the position of $\mathbf{P}_i$ causes the shape of the curve segment defined on $[u_i, u_{i+p+1})$ to change (*i.e.*, the local modification properly). Thus, modifying control point $\mathbf{P}_5$ changes the shape of the curve segment defined on $[u_5, u_{10}) = [1/4, 1)$. ∎

2. **NURBS**

(a) **[10 points]** Suppose we have a NURBS curve of degree 4 defined by control points P_0, P_1, ..., P_{20} and weights $w_0, w_1, \ldots, w_{20}$. If the weight of point P_{10} is changed, which part of the NURBS curve will be affected? List the interval(s) on which the affected curve segment is defined, and elaborate your claim. **You will receive no credit if you do not provide a correct elaboration.**

Answer: Since a 3D NURBS curve is the central projection of a 4D B-spline curve, if a segment of the B-spline curve does not change shape, the corresponding curve segment of the projection NURBS curve does not change shape either. Let $\mathbf{P}_i^w$ be the "lifted" 4D control point of control point $\mathbf{P}_i$. Since modifying w_{10} only changes the position of $\mathbf{P}_{10}^w$, the curve segment of the 4D B-spline defined on $[u_{10}, u_{15})$ changes its shape because of the local modification property. As a result, only the curve segment defined on $[u_{10}, u_{15})$ of the NURBS curve will be affected. ∎

(b) Suppose we have a NURBS curve $\mathbf{C}(u)$ of degree 2 defined by control points $(-2, 0)$, $(0, 4)$ and $(2, 0)$ with weights 1, 3 and 1, and knots 0, 0, 0, 1, 1, 1. Do the following:

i. **[25 points]** Compute $\mathbf{C}(0.5)$ using de Boor's algorithm. **You will receive no credit if you only provide an answer without all calculation steps.**

Answer: Since a 3D NURBS curve is the projection of a 4D B-spline curve, we can compute the point on the 4D B-spline curve and project it back to the 3D NURBS curve. The 2D control points and their weights are $(-2, 0)$ with weight 1, $(0, 4)$ with weight 3, and $(2, 0)$ with weight 1. The first step is the use of homogeneous coordinates by adding 1 to the third coordinate, yielding: $(-2, 0, 1)$, $(0, 4, 1)$ and $(2, 0, 1)$. The 3D control points for the B-spline curve are computed by multiplying each control point in homogeneous form by its weight: $(-2, 0, 1)$, $(0, 12, 3)$ and $(2, 0, 1)$.

The point at $u = 0.5$ on this 3D B-spline curve can be easily computed with de Boor's algorithm. Since this is a B-spline curve of degree 2 with knots $\{0, 0, 0, 1, 1, 1\}$, it is actually a Bézier curve of degree 2 defined by the same set of control points, and, as a result, de Casteljau's algorithm (rather than de Boor's algorithm) can be used. The following shows the de Casteljau's algorithm computation steps:

$$
\begin{array}{ccccc}
(-2,0,1) & & & & \\
 & \searrow & & & \\
 & & (-1,6,2) & & \\
 & \nearrow & & \searrow & \\
(0,12,3) & & & & (0,6,2) \\
 & \searrow & & \nearrow & \\
 & & (1,6,2) & & \\
 & \nearrow & & & \\
(2,0,1) & & & &
\end{array}
$$

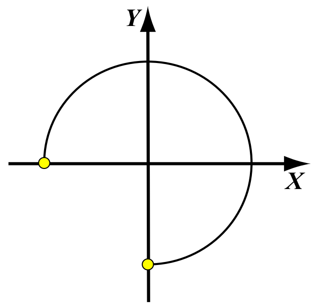
Therefore, converting the homogeneous coordinates of $(0, 6, 2)$ back to 2D Euclidean coordinates yields the point on the 2D NURBS curve $(0, 3)$. ∎

ii. **[10 points]** Divide the NURBS into two at $u = 0.5$. What are the control points, their weights, and knots?

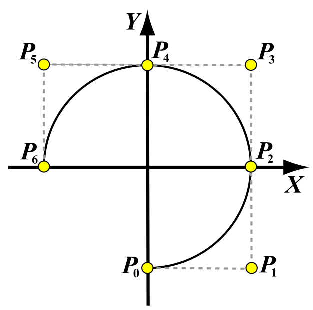
Answer: The 3D B-spline control points $(-2, 0, 1)$, $(-1, 6, 2)$ and $(0, 6, 2)$ project back to 2D to define the "left" NURBS curve segment. Thus, the control points and weights that define

the NURBS curve segment on $[0, 0.5]$ are $(-2, 0)$ with weight 1, $(-1/2, 3)$ with weight 2, and $(0, 3)$ with weight 2. The knot vector is $\{0, 0, 0, 0.5, 0.5, 0.5\}$.
Similarly, the "right" NURBS curve segment on $[0.5, 1]$ are defined by $(0, 3)$ with weight 2, $(1/2, 3)$ with weight 2 and $(2, 0)$ with weight 1. The knot vector is $\{0.5, 0.5, 0.5, 1, 1, 1\}$. ■

(c) **[10 points]** Define a NURBS curve of three quarters of a complete circle of radius 1 as shown below. More precisely, the center is at the original, the beginning point of this 3-quarter circle is $(0, -1)$ and the ending point is $(-1, 0)$. **An answer is not enough. You have to show how and why you find your answer,**



__**Answer**__: Since each quarter circle requires three control points, three quarter circles require seven control points $\mathbf{P}_0, \mathbf{P}_1, \ldots, \mathbf{P}_6$ (*i.e.*, $n = 6$) as shown below, where the control points in the desired order are $\mathbf{P}_0 = (0, -1)$, $\mathbf{P}_1 = (1, -1)$, $\mathbf{P}_2 = (1, 0)$, $\mathbf{P}_3 = (1, 1)$, $\mathbf{P}_4 = (0, 1)$, $\mathbf{P}_5 = (-1, 1)$ and $\mathbf{P}_6 = (-1, 0)$:



Since $m = n+p+1$ and since $p = 2$, $m = 9$ and 10 knots are needed. Since the circle is "clamped" at $\mathbf{P}_0$ and $\mathbf{P}_6$, we have $u_0 = u_1 = u_2 = 0$ and $u_7 = u_8 = u_9 = 1$. Based on the way of defining a complete circle, since this 3-quarter circle is tangent to the control polygon at $\mathbf{P}_2$ and $\mathbf{P}_4$, the other four knots are $u_3 = u_4 = 1/3$ and $u_5 = u_6 = 2/3$. Finally, the weights of $\mathbf{P}_1$, $\mathbf{P}_3$ and $\mathbf{P}_5$ are $\sqrt{2}/2$, and the weights of the remaining control points are set to 1. ■