CS5311 Homework #1 (Spring 2001)

Due: Thursday, January 31, 2002 **before class**

100 points

You can only discuss the meaning of a problem with each other. Any forms of collaboration are absolutely forbidden. I will forward any suspicious cases to the Dean of Students or Graduate School.

When you write your program, make sure you also supply an explanation of your program, preferably a general elaboration followed by an instruction-by-instruction discussion. Nonsense, or direct translations of instructions (*i.e.*, add 1 to X), is unacceptable.

1. **[6 points]** Suppose we have an infinite number of sets S_0, S_1, S_2, S_3, ..., each of which satisfies a property $\mathcal{P}$. Moreover, we also know that the union of two sets that satisfy property $\mathcal{P}$ also satisfies property $\mathcal{P}$. The following induction proves that $A = \bigcup_{i \geq 0} S_i$ satisfies property $\mathcal{P}$.

   (a) **Base Step $n = 0$ and $n = 1$:** Since $A = S_0$ and S_0 satisfies property $\mathcal{P}$, so does A. For $n = 1$, we have $A = S_0 \cup S_1$ and A satisfies property $\mathcal{P}$.

   (b) **Induction Step:** Suppose the proposition holds for n. Then, we have $A = S_0 \cup S_1 \cup \cdots \cup S_n \cup S_{n+1} = (S_0 \cup S_1 \cup \cdots \cup S_n) \cup S_{n+1}$. By the induction hypothesis, $B = S_0 \cup S_1 \cup \cdots \cup S_n$ satisfies property $\mathcal{P}$. Because B and S_{n+1} satisfy property $\mathcal{P}$, their union $A = B \cup S_{n+1} = S_0 \cup S_1 \cup \cdots \cup S_n \cup S_{n+1}$ satisfies property $\mathcal{P}$

   (c) From the induction, since the proposition holds for every $n \geq 0$, it holds for $A = \bigcup_{i \geq 0} S_i$, and A satisfies property $\mathcal{P}$.

   Is this proof completely correct? If it is not completely correct, correct the mistakes to make this proof work. If it is completely incorrect, point out the problem and elaborate why it is a problem.

2. (a) **[4 points]** Given a constant $k \geq 0$, show that the constant function $f_k(x) = k$ is computable.

   (b) **[10 points]** A *straightline program* is a program that does not contain the IF-GOTO instruction. Show that if the length of a straightline program $\mathcal{P}$ is k, then $\psi_{\mathcal{P}}^{(1)}(x) \leq k$ for all k. This means the output of a straightline program of k instructions must be less than or equal to k. Note that we have only three types of instructions $V \leftarrow V + 1$, $V \leftarrow V - 1$ and IF $V \neq 0$ GOTO L. (**Hint:** use induction on program length k)

   (c) **[6 points]** Show that if $\mathcal{P}$ is a program that computes f_k, then the length of $\mathcal{P}$ is least k.

   (d) **[6 points]** Show that no straightline program can compute the function $g(x) = x + 1$. Therefore, the class of all straightline programs is a proper subset of the class of all computable functions.

3. Let COMP be the class of functions obtained from the initial functions by a finite sequence of compositions.

   (a) **[15 points]** Show that every function $f(x_1, x_2, \ldots, x_n)$ in COMP, either it is a constant function $f(x_1, x_2, \ldots, x_n) = k$ for some constant k, or $f(x_1, x_2, \ldots, x_n) = x_i + k$ for some $1 \leq i \leq n$ and some constant k.

(b) **[6 points]** An n-ary function f is *monotonic* if for all n-tuples $(x_1, x_2, \ldots, x_n)$ and $(y_1, y_2, \ldots, y_n)$ such that $x_i \leq y_i$ for all $1 \leq i \leq n$, then $f(x_1, x_2, \ldots, x_n) \leq f(y_1, y_2, \ldots, y_n)$. Show that every function in COMP is monotonic.

(c) **[6 points]** A predicate is *trivial* if and only if it is always TRUE or always FALSE. Show that no nontrivial predicate can be a member of COMP. Therefore, COMP is proper subset of the class of primitive recursive functions.

4. Prove or disprove the following propositions:

   (a) **[8 points]** The class of all total functions is a PRC class.

   (b) **[8 points]** The intersection of two PRC classes is a PRC class.

5. **[10 points]** Prove or disprove: if function $f(x_1, x_2, \ldots, x_n)$ is a primitive recursive function, then function $g(x) = f(\underbrace{x, x, \ldots, x}_{n \text{ terms}})$ is also primitive recursive.

6. **[15 points]** Suppose function f is defined as follows:

$$f(x, 0) = g(x) \qquad \text{and} \qquad f(x, y+1) = f(f(x, y), y)$$

   where g is a primitive recursive function. Show that f is primitive recursive.

Print this page and attach it as the last page of your submission

Grade Report

Problem		*Assigned*	*Received*
1		6	
2	a	4	
	b	10	
	c	6	
	d	6	
3	a	15	
	b	6	
	c	6	
4	a	8	
	b	8	
5		10	
6		15	
Total		100	