CS5311 Homework #3 Solutions (Spring 2002)

Due: Thursday, March 21, 2002 before class

100 points

1. [**5 points**] What is wrong with the following diagonalization proof?

   - We show that there is a partially computable function that is not in the list Φ_0, Φ_1, Φ_2, ..., Φ_i, ...
   - Let $f(n) = 1 + \Phi(n, n) = 1 + \Phi_n(n)$
   - Then, f is partially computable
   - Since f is not equal to Φ_n for every n, f is not in the list.

   Answer: It is possible that $\Phi(n, n)$ is not defined for some n. Thus, $f(n) = 1 + \Phi(n, n)$ does not imply $f(n) \neq \Phi(n, n)$, and the diagonalization argument fails. ■

2. [**10 points**] Suppose g is a computable function. Show that there exists a computable function f such that for all x, $W_{f(x)} = W_x \cup \{g(x)\}$. (**Hint:** Use the s-m-n Theorem to construct f).

   Answer: The proposition states that an element y is in $W_{f(x)}$ if and only if y is in W_x or y is equal to $g(x)$. This is reflected in the following construction:

$$h(y, x) = \begin{cases} 1 & y \in W_x \vee y = g(x) \\ \uparrow & \text{otherwise} \end{cases}$$

   This is a partially computable function. Why? Let $\#h$ be the program number that computes h. By the s-m-n Theorem, we have $\Phi(y, x, \#h) = \Phi(y, S_1^1(x, \#h)) = \Phi(y, f(x))$, where $f(x) = S_1^1(x, \#h)$. Since $f(x)$ is primitive recursive, it is computable. Then, we have

$$\begin{aligned} y \in W_{f(x)} &\Leftrightarrow \Phi(y, f(x))\downarrow \\ &\Leftrightarrow \Phi(y, x, \#h)\downarrow \\ &\Leftrightarrow h(y, x)\downarrow \\ &\Leftrightarrow y \in W_x \ \vee \ y = g(x) \end{aligned}$$

   This means the elements of $W_{f(x)}$ consists of those of W_x or $g(x)$. Hence, $W_{f(x)} = W_x \cup \{\ g(x)\ \}$. ■

3. [**10 points**] Show that if A is a r.e. set, then $B = \cup_{x \in A} W_x$ is also a r.e. set. (**Hint:** Use the Projection Theorem)

   Answer: Since A is r.e., there exists an n such that $A = W_n$. Then, we have

$$\begin{aligned} b \in B &\Leftrightarrow (\exists x)[x \in A \ \wedge \ b \in W_x] \quad (B\text{'s definition}) \\ &\Leftrightarrow (\exists x)(\exists n)[x \in W_n \ \wedge \ b \in W_x] \\ &\Leftrightarrow (\exists x)(\exists n)[\mathrm{HALT}(n, x) \ \wedge \ \mathrm{HALT}(x, b)] \\ &\Leftrightarrow (\exists x)(\exists n)[(\exists u)[\mathrm{STEP}^{(1)}(x, n, u)] \ \wedge \ (\exists v)[\mathrm{STEP}^{(1)}(b, x, v)]] \end{aligned}$$

The above has four numbers involved x, n, u and v. We can encode them into one number using Gödel numbering. More precisely, let t be $[x, n, u, v]$. The above is rewritten as follows:

$$b \in B \Leftrightarrow (\exists t)[\text{Len}(t) = 4 \ \wedge \ \text{STEP}^{(1)}((t)_1, (t)_2, (t)_3)] \ \wedge \ \text{STEP}^{(1)}(b, (t)_1, (t)_4)]$$

Define predicate $R(b, t)$ to be

$$R(b, t) \equiv \text{Len}(t) = 4 \ \wedge \ \text{STEP}^{(1)}((t)_1, (t)_2, (t)_3)] \ \wedge \ \text{STEP}^{(1)}(b, (t)_1, (t)_4)$$

Therefore, $b \in B \Leftrightarrow (\exists t)[R(b, t)]$. Since $R(b, t)$ is primitive recursive, B is r.e. by the Projection Theorem. ∎

4. [**10 points**] Show that $A = \{ \ n \mid \{0, 1, \ldots, n\} \subseteq W_n \ \}$ is r.e. (**Hint:** Use the Projection Theorem)

Answer: From the definition of A, we have the following:

$$\begin{aligned} n \in A \ &\Leftrightarrow \ \{0, 1, \ldots, n\} \subseteq W_n \\ &\Leftrightarrow \ \Phi(0, n)\downarrow \ \wedge \ \Phi(1, n)\downarrow \ \wedge \ \cdots \ \wedge \ \Phi(n, n)\downarrow \\ &\Leftrightarrow \ (\exists t_0)[\text{STEP}^{(1)}(0, n, t_0)] \ \wedge \ (\exists t_1)[\text{STEP}^{(1)}(1, n, t_1)] \ \wedge \ \cdots \ \wedge \ (\exists t_n)[\text{STEP}^{(1)}(n, n, t_n)] \end{aligned}$$

Let $t = [t_0, t_1, \ldots, t_n]$. The above becomes

$$\begin{aligned} n \in A \ &\Leftrightarrow \ (\exists t)[\text{Len}(t) = n \ \wedge \ \text{STEP}^{(1)}(0, n, (t)_0) \ \wedge \ \text{STEP}^{(1)}(1, n, (t)_1) \ \wedge \ \cdots \ \wedge \ \text{STEP}^{(1)}(n, n, (t)_n)] \\ &\Leftrightarrow \ (\exists t)[\text{Len}(t) = n \ \wedge \ (\forall_{i \leq n})[\text{STEP}^{(1)}(i, n, (t)_i)]] \end{aligned}$$

Define $R(n, t)$ as follows:

$$R(n, t) \equiv \text{Len}(t) = n \ \wedge \ (\forall_{i \leq n})[\text{STEP}^{(1)}(i, n, (t)_i)]$$

Since $R(n, t)$ is primitive recursive (why?), by the Projection Theorem, set A is r.e. ∎

5. [**10 points**] Assume that A and B are both nonempty *proper* subsets of $\mathcal{N}$. Prove that if $A - B$ and $B - A$ are both recursive, then $A \leq_{\text{m}} B$.

Answer: Since B is non-empty, there is a $b \in B$. Since B is a proper subset of $\mathcal{N}$, there is a $\bar{b} \in B$. Define function f as follows:

$$f(x) = \begin{cases} b & \text{if } x \in A - B \\ \bar{b} & \text{if } x \in B - A \\ x & \text{otherwise} \end{cases}$$

Since $A - B$ and $B - A$ are recursive, the membership testings for $A - B$ and $B - A$ are computable. That is, $x \in A - B$ and $x \in B - A$ are computable. Consequently, f is computable. Now, we have

$$\begin{aligned} x \in A \ &\Rightarrow \ (x \in A - B) \ \vee \ (x \in A \cap B) \qquad (\text{since } A = (A - B) \cup (A \cap B)) \\ &\Rightarrow \ (f(x) = b) \ \vee \ (f(x) = x \in A \cap B)) \\ &\Rightarrow \ f(x) \in B \end{aligned}$$

On the other hand, we also have

$$\begin{aligned} x \notin A &\Rightarrow (x \in B - A) \ \vee \ (x \in \overline{A \cup B}) \\ &\Rightarrow (f(x) = \bar{b} \notin B) \ \vee \ (f(x) = x \notin B) \\ &\Rightarrow f(x) \notin B \end{aligned}$$

Therefore, we have $x \in A \Leftrightarrow f(x) \in B$, and $A \leq_{\mathrm{m}} B$. ∎

6. [**15 points**] Let FINITE $= \{ x \mid x \in \mathcal{N}$ and W_x is finite $\}$. Show that $\bar{K} \leq_{\mathrm{m}}$ FINITE. What does this reduction mean?

   Answer: To perform this reduction, we need to find a computable function f such that $x \in \bar{K} \Leftrightarrow f(x) \in$ FINITE. Consider the following function:

$$g(y, x) = \begin{cases} y & \text{if } x \in K \\ \uparrow & \text{otherwise} \end{cases}$$

   Thus, if $x \in K$, function $g(y, x)$ in y is defined everywhere and hence is not in FINITE. On the other hand, if $x \in \bar{K}$, function $g(y, x)$ in y is defined nowhere, and, is in FINITE. Here is a formal proof. Since $g(y, x)$ is partially computable (why?), there is a program $\#g$ that computes g. By the *s-m-n* Theorem, we have $\Phi^{(2)}(y, x, \#g) = \Phi^{(1)}(y, S_1^1(x, \#g)) = \Phi^{(1)}(y, f(x))$, where $f(x) = S_1^1(x, \#g)$. Then, we have

$$\begin{aligned} x \in \bar{K} &\Rightarrow (\forall_y)[g(y, x)\uparrow] \\ &\Rightarrow (\forall_y)[\Phi^{(1)}(y, x, \#g)\uparrow] \\ &\Rightarrow (\forall_y)[\Phi^{(1)}(y, S_1^1(x, \#g))\uparrow] \\ &\Rightarrow (\forall_y)[\Phi^{(1)}(y, f(x))\uparrow] \\ &\Rightarrow f(x) \text{ is defined nowhere} \\ &\Rightarrow W_{f(x)} = \emptyset \\ &\Rightarrow f(x) \in \text{FINITE} \end{aligned}$$

   On the other hand, we also have

$$\begin{aligned} x \notin \bar{K} &\Rightarrow x \in K \\ &\Rightarrow (\forall_y)[g(y, x) = y] \\ &\Rightarrow (\forall_y)[\Phi^{(1)}(y, f(x)) = y] \\ &\Rightarrow f(x) \text{ is the identity function} \\ &\Rightarrow W_{f(x)} = \mathcal{N} \\ &\Rightarrow f(x) \notin \text{FINITE} \end{aligned}$$

   Therefore, $x \in \bar{K} \Leftrightarrow f(x) \in$ FINITE and $\bar{K} \leq_{\mathrm{m}}$ FINITE.

   This reduction shows that FINITE is not r.e., which also means there is no program whatsoever (computable or partially computable) that can test if an arbitrary program is defined on a finite number of input values. ∎

7. **[15 points]** Let TOTAL $= \{ x \mid \Phi_x \text{ is total} \}$. Show that $\bar{K} \leq_{\text{m}}$ TOTAL.

   Answer: Consider the following function:

$$g(y,x) = \begin{cases} 1 & \neg\text{STEP}(x,x,y) \\ \uparrow & \text{STEP}(x,x,y) \end{cases}$$

   Thus, g is a partially computable function. Let $\#g$ be its program number. By the *s-m-n* Theorem, we have $\Phi^{(2)}(y,x,\#g) = \Phi^{(1)}(y, S_1^1(x,\#g)) = \Phi^{(1)}(y, f(x))$, where $f(x) = S_1^1(x,\#g)$. Then, we have

$$\begin{aligned} x \in \bar{K} &\Rightarrow \Phi(x,x)\uparrow \\ &\Rightarrow (\forall_y)[\neg\text{STEP}(x,x,y)] \\ &\Rightarrow (\forall_y)[g(y,x) = 1] \\ &\Rightarrow (\forall_y)[\Phi(y,f(x)) = 1] \\ &\Rightarrow f(x) \text{ is a constant function} \\ &\Rightarrow f(x) \in \text{TOTAL} \end{aligned}$$

   Conversely, we have

$$\begin{aligned} x \notin \bar{K} &\Rightarrow x \in K \\ &\Rightarrow \Phi(x,x)\downarrow \\ &\Rightarrow (\exists_y)[\text{STEP}(x,x,y)] \\ &\Rightarrow (\exists_y)[g(y,x)\uparrow] \\ &\Rightarrow (\exists_y)[\Phi(y,f(x))\uparrow] \\ &\Rightarrow f(x) \text{ is not total} \\ &\Rightarrow f(x) \notin \text{TOTAL} \end{aligned}$$

   Therefore, we have $x \in \bar{K} \Leftrightarrow f(x) \in$ TOTAL, and $\bar{K} \leq_{\text{m}}$ TOTAL. ∎

8. **[15 points]** Let TOTAL $= \{ x \mid \Phi_x \text{ is total} \}$. Show that TOTAL is not recursive using the Recursion Theorem.

   Answer: If TOTAL is recursive, $x \in$ TOTAL is computable. Consider the following function:

$$g(y,x) = \begin{cases} \uparrow & y \in \text{TOTAL} \\ 1 & \text{otherwise} \end{cases}$$

   Therefore, $g(y,x)$ is partially computable. Why? By the Recursion Theorem, there is an e such that $\Phi(x,e) = g(e,x)$ holds. Is e in TOTAL?

$$\begin{aligned} e \in \text{TOTAL} &\Rightarrow (\forall_x)[g(e,x)\uparrow] \\ &\Rightarrow (\forall_x)[\Phi(x,e)\uparrow] \\ &\Rightarrow e \text{ is not total} \\ &\Rightarrow e \notin \text{TOTAL} \end{aligned}$$

On the other hand, we have

$$\begin{aligned}
e \notin \text{TOTAL} &\Rightarrow (\forall_x)[g(e,x) = 1] \\
&\Rightarrow (\forall_x)[\Phi(x,e) = 1] \\
&\Rightarrow e \text{ is a constant function} \\
&\Rightarrow e \text{ is total} \\
&\Rightarrow e \in \text{TOTAL}
\end{aligned}$$

Hence, we have $e \in \text{TOTAL} \Leftrightarrow e \notin \text{TOTAL}$, which is a contradiction, and TOTAL is not recursive. ∎

9. [**10 points**] Show that there is an n satisfying $W_n = K$ and $n \in K$. (**Hint:** Use the Recursion Theorem)

   Answer: Consider the following function:

$$g(y,x) = \begin{cases} 1 & x, y \in K \\ \uparrow & \text{otherwise} \end{cases}$$

   This is a partially computable function. By the Recursion Theorem, there is a n such that $\Phi(x,n) = g(n,x)$. Now, we study the set on which n is defined:

$$\begin{aligned}
x \in W_n &\Leftrightarrow \Phi(x,n)\downarrow \\
&\Leftrightarrow g(n,x)\downarrow \\
&\Leftrightarrow x, n \in K \\
&\Leftrightarrow n \in K \text{ and } x \in K
\end{aligned}$$

   Thus, W_n contains all elements in K because $x \in K$. Moreover, n is in K. Therefore, we have $W_n = K$ and $n \in K$. ∎